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# Density of states and thermodynamic properties of an ideal system trapped in any dimension

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**Abstract.** The density of states and thermodynamic properties of an ideal gas system trapped in a generic power-law potential in an $n$-dimensional space are studied. A unified description for the Bose, Fermi and classical gases is given by using the grand potential of the system. Consequently, not only the results in current textbooks of statistical mechanics but also some new general important conclusions, such as the conditions for the occurence of Bose–Eisntein condensation in a trapped Bose system in any dimensional space, can be derived directly using the results of this paper.

## 1. Introduction

It is very important to calculate the density of states of particles when the method of statistical physics is used to investigate the thermodynamic properties of some systems. In general textbooks and literature [1, 2], one mainly calculates the density of states of free particles (i.e. in the absence of any external potential). However, real systems are often trapped in external potentials with various different shapes, and their densities of states are very different from those of free particles. For example, Bose–Einstein condensation (BEC), first observed in 1995 [3–5], was just an outcome of the mangetic trap and cooling techniques used to realize the constrained role of the external potential for Bose atomic gases. Thus, it is necessary to investigate the density of states of a system trapped in external potentials.

Although the interaction between the particles is extremely important in a real system, the problems are made tractable and the essential physics is retained by assuming an ideal system of non-interacting particles. It was pointed out [6] that the influence of the finite number of particles and their interaction on the condensation temperature of BEC is several per cent, so that it is a good approximation to treat such systems as ideal ones.

In this paper, we consider an ideal gas trapped in a generic power-law potential in an $n$-dimensional space. A general expression for the density of states is derived. Starting from the grand potential of the system, we give a unified description for the thermodynamic properties of degenerate (Bose and Fermi) and non-degenerate systems.

This paper is organized as follows. In section 2 we derive the general expressions of the density of states and the grand potential of an ideal system. The thermodynamic quantities of the system are given in terms of the grand potential. In sctions 3 and 4 we analyse the thermodynamic properties of degenerate Bose and Fermi systems, respectively.

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In section 5 the thermodynamic properties of weakly degenerate and non-degenerate systems are straightforwardly derived from the above results. In section 6 the relation between the density of states and the phase transition is determined. Finally, all the important results are summarized and a short discussion given in section 7.

## 2. The density of states and thermodynamic quantities of an ideal system

We consider an ideal system trapped in a generic power-law potential in an $n$-dimensional space with a single-particle Hamiltonian

$$H = \varepsilon_0 \left(\frac{p}{p_0}\right)^s + \sum_{i=1}^{n} U_i \left|\frac{r_i}{L_i}\right|^{t_i} \tag{1}$$

where $\varepsilon_0$, $p_0$, $s$, $U_i$, $L_i$, and $t_i$ are all positive constants, and $p$ and $r_i$ are the momentum and the components of coordinate of a particle, respectively.

When the number of particles in the system is large and the potential energy of particles in a trap is much smaller than their kinetic energy (this condition is often satisfied), the Thomas–Fermi semiclassical approximation is valid [7]. Thus sums over quantum states may be replaced by integrals over phase space. The total number of quantum states for $H \leqslant \varepsilon$ may be expressed as

$$\Sigma(\varepsilon) = \frac{g}{h^n} \int_{H\leqslant\varepsilon} \prod_{i=1}^{n} (\mathrm{d}r_i \, \mathrm{d}p_i) \tag{2}$$

where $h$ is Planck's constant and $g$ is the spin degenerate factor. From the expression of the volume of an $n$-dimensional sphere $V(n, R) = C_n R^n = [\pi^{n/2}/\Gamma(n/2+1)]R^n$, one can obtain

$$\mathrm{d}V(n, R) = S(n, R)\,\mathrm{d}R = nC_n R^{n-1}\,\mathrm{d}R \tag{3}$$

where $S(n, R)$ is the surface area of the $n$-dimensional sphere. By using equation (3) and the Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

equation (2) may be expressed as

$$\begin{aligned}
\Sigma(\varepsilon) &= \frac{g}{h^n} \int \prod_{i=1}^{n} \mathrm{d}r_i \int_0^{p_1} nC_n p^{n-1}\,\mathrm{d}p = \frac{gC_n}{h^n} \int \prod_{i=1}^{n} (p_1)^n\,\mathrm{d}r_i \\
&\equiv \frac{gC_n p_0^n}{h^n \varepsilon_0^{n/s}} \varepsilon^{n/s} \left(\prod_{i=1}^{n} \frac{L_i \varepsilon^{1/t_i}}{U_i^{1/t_i}}\right) \int \left[1 - \sum_{i=1}^{n} |Y_i|^{t_i}\right]^{n/s} \prod_{i=1}^{n} \mathrm{d}Y_i \\
&\equiv \frac{gC_n p_0^n}{h^n \varepsilon_0^{n/s}} \varepsilon^{\lambda} \left(\prod_{i=1}^{n} \frac{L_i}{U_i^{1/t_i}}\right) F(t_1, \ldots, t_n) \\
&\equiv \alpha \frac{\Gamma(n/s+1)}{\Gamma(\lambda+1)} \varepsilon^{\lambda}
\end{aligned} \tag{4}$$

where

$$\alpha = \frac{g2^n C_n}{h^n} \frac{p_0^n}{\varepsilon_0^{n/s}} \prod_{i=1}^{n} \frac{L_i \Gamma(1/t_i + 1)}{U_i^{1/t_i}} \tag{5}$$

$$\lambda = \frac{n}{s} + \sum_{i=1}^{n} \frac{1}{t_i} \tag{6}$$

$\Gamma(l) = \int_0^\infty y^{l-1} e^{-y}\, dy$ is the Gamma function,

$$p_1 = \left[\varepsilon - \sum_{i=1}^{n} U_i \left|\frac{r_i}{L_i}\right|^{t_i}\right]^{1/s} \frac{p_0}{\varepsilon_0^{1/s}}$$

$$|Y_i|^{t_i} = \frac{U_i}{\varepsilon}\left|\frac{r_i}{L_i}\right|^{t_i}$$

$$\begin{aligned} F(t_1, \ldots, t_n) = & \left[\int_{-1}^{1} (1 - |X_1|^{t_1})^{(n/s)+\sum_{i=2}^{n} 1/t_i}\, dX_1\right] \\ & \times \left[\int_{-1}^{1} (1 - |X_2|^{t_2})^{(n/s)+\sum_{i=3}^{n} 1/t_i}\, dX_2\right] \\ & \cdots \left[\int_{-1}^{1} (1 - |X_{n-1}|^{t_{n-1}})^{(n/s)+1/t_n}\, dX_{n-1}\right] \\ & \times \left[\int_{-1}^{1} (1 - |X_n|^{t_n})^{n/s}\, dX_n\right] \\ = & \frac{2^n \Gamma(n/s + 1)}{\Gamma(\lambda + 1)} \prod_{i=1}^{n} \Gamma\left(\frac{1}{t_i} + 1\right) \end{aligned}$$

and

$$|X_i|^{t_i} = \frac{|Y_i|^{t_i}}{\left[1 - \sum_{j=1}^{i-1} |Y_j|^{t_j}\right]}.$$

The derivative of equation (4) with respect to $\varepsilon$ yields the density of states as

$$D(\varepsilon) = \frac{\partial \Sigma(\varepsilon)}{\partial \varepsilon} = \alpha \frac{\Gamma(n/s + 1)}{\Gamma(\lambda)} \varepsilon^{\lambda - 1}. \tag{7}$$

For the case of a spherical symmetric potential, the Hamiltonian of a single particle $H = \varepsilon_0 (p/p_0)^s + U_0 (r/L_0)^t$. By using the similar method mentioned above, the total number of quantum states and the density of states can respectively be expressed by

$$\Sigma(\varepsilon) = \alpha_0 \frac{\Gamma(n/s + 1)}{\Gamma(\lambda_0 + 1)} \varepsilon^{\lambda_0} \tag{8}$$

and

$$D(\varepsilon) = \alpha_0 \frac{\Gamma(n/s + 1)}{\Gamma(\lambda_0)} \varepsilon^{\lambda_0 - 1} \tag{9}$$

where

$$\alpha_0 = \frac{g C_n^2}{h^n} \frac{p_0^n}{\varepsilon_0^{n/s}} \frac{L_0^n}{U_0^{n/t}} \Gamma(n/t + 1) \tag{10}$$

and

$$\lambda_0 = n/s + n/t. \tag{11}$$

This is non-trivial. Only if $n = 1$, $t_i = 2$, or $t_i \to \infty$ can equation (7) be reduced to (9). For other values of $n$ and $t_i$, equation (9) cannot be deduced from (7).

Now, we assume that the system may be described by a grand canonical ensemble. The grand potential is defined by

$$q(z,\beta,\alpha) = q_0 + \frac{1}{b}\int_0^\infty \ln\left(1 + bz\mathrm{e}^{-\beta\varepsilon}\right) D(\varepsilon)\,\mathrm{d}\varepsilon \tag{12}$$

where $\beta = 1/kT$, $k$ is Boltzmann's constant, $T$ is the absolute temperature, $z = \exp(\mu/kT)$ is the fugacity, $\mu$ is the chemical potential, $b$ is equal to $-1$, 1 and 0 for the Bose, Fermi, and classical systems, respectively, and $q_0 = (1/b)\ln(1 + bz)$. Substituting equation (7) into (12), one can obtain

$$q(z,\beta,\alpha) = q_0 + \frac{\alpha}{b}\frac{\Gamma(n/s+1)}{\Gamma(\lambda+1)}\int_0^\infty \ln\left(1 + bz\mathrm{e}^{-\beta\varepsilon}\right)\mathrm{d}\varepsilon^\lambda. \tag{13}$$

By using integration by parts, equation (13) may be expressed in the following form:

$$\begin{aligned} q &= q_0 + \alpha\beta\frac{\Gamma(n/s+1)}{\Gamma(\lambda+1)}\int_0^\infty \frac{1}{z^{-1}\mathrm{e}^{\beta\varepsilon} + b}\varepsilon^\lambda\,\mathrm{d}\varepsilon \\ &= q_0 + \alpha\Gamma(n/s+1)(kT)^\lambda \times \begin{cases} g_{\lambda+1}(z) & \text{Bose system} \\ f_{\lambda+1}(z) & \text{Fermi system} \\ z & \text{classical system} \end{cases} \\ &\equiv q_0 + NB \end{aligned} \tag{14}$$

where

$$N = z\left[\frac{\partial q}{\partial z}\right]_{\beta,\alpha} = N_0 + \alpha\Gamma(n/s+1)(kT)^\lambda \times \begin{cases} g_{\lambda}(z) & \text{Bose system} \\ f_{\lambda}(z) & \text{Fermi system} \\ z & \text{classical system} \end{cases} \tag{15}$$

is the total number of particles in the system,

$$N_0 = z\left[\frac{\partial q_0}{\partial z}\right]_{\beta,\alpha} = \frac{z}{1 + bz} \tag{16}$$

is the ground state occupation which is equal to zero compared with $N$ except in the case of $z = 1$ in the Bose systems,

$$B = \begin{cases} (1 - N_0/N)g_{\lambda+1}(z)/g_\lambda(z) & \text{Bose system} \\ f_{\lambda+1}(z)/f_\lambda(z) & \text{Fermi system} \\ 1 & \text{classical system} \end{cases} \tag{17}$$

and

$$g_x(z) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{y^{x-1}\,\mathrm{d}y}{z^{-1}\mathrm{e}^y - 1} = \sum_{j=1}^\infty \frac{z^j}{j^x} \tag{18}$$

$$f_x(z) = \frac{1}{\Gamma(x)}\int_0^\infty \frac{y^{x-1}\,\mathrm{d}y}{z^{-1}\mathrm{e}^y + 1} = \sum_{j=1}^\infty (-1)^{j-1}\frac{z^j}{j^x} \tag{19}$$

are the Bose and Fermi integrals, respectively.

Using the statistical expressions of the thermodynamic quantities, one can easily obtain the thermodynamic quantities of the system from equation (14). For instance, the total energy $E$ and entropy $S$ of the system are respectively given by

$$E = -\left[\frac{\partial q}{\partial \beta}\right]_{z,\alpha} = NkT\lambda B \tag{20}$$

and

$$S = k(q - N \ln z + \beta E) = Nk[B(\lambda + 1) - \ln z]. \tag{21}$$

## 3. Thermodynamic properties of a degenerate Bose gas

The fugacity $z$ of a Bose system is decided by the characteristics of the distribution function. Its value must be restricted to the range $0 < z < 1$ and increases monotonically as the temperature decreases. When $T$ is equal to the critical temperature $T_c$, $z = 1$. When $T < T_c$, $z = 1$ is kept constant and the Bose integral becomes the Riemann Zeta function, i.e. $g_x(1) = \zeta(x)$. In such a case, according to $N_0 = z/(1 - z)$, there is a macroscopic quantity of particles $N_0$ with zero energy condensed into the ground state, while the number of particles in the excited states decreases continuously as temperature decreases. This is the well known Bose–Einstein condensation.

Using equations (15), (20) and (21), we obtain

$$N = \begin{cases} \alpha\Gamma(n/s + 1)(kT)^{\lambda} g_{\lambda}(z) & T > T_c \\ N_0 + \alpha\Gamma(n/s + 1)(kT)^{\lambda}\zeta(\lambda) & T \leqslant T_c \end{cases} \tag{22}$$

$$\frac{E}{N} = \lambda kT \begin{cases} g_{\lambda+1}(z)/g_{\lambda}(z) & T > T_c \\ (1 - N_0/N)\zeta(\lambda + 1)/\zeta(\lambda) & T \leqslant T_c \end{cases} \tag{23}$$

and

$$\frac{S}{Nk} = \begin{cases} (\lambda + 1)g_{\lambda+1}(z)/g_{\lambda}(z) - \ln z & T > T_c \\ (1 - N_0/N)(\lambda + 1)\zeta(\lambda + 1)/\zeta(\lambda) & T \leqslant T_c. \end{cases} \tag{24}$$

From equation (22), we can find that the critical temperature $T_c$ and the fraction of condensation $N_0/N$ are respectively determined by

$$T_c = \frac{1}{k}\left[\frac{N}{\alpha\Gamma(n/s + 1)\zeta(\lambda)}\right]^{1/\lambda} \tag{25}$$

and

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{\lambda}. \tag{26}$$

It should be noted that when $T > T_c$, $z$ is a function of temperature, $\partial g_{x+1}(z)/\partial \ln z = g_x(z)$, and $\partial N/\partial T = 0$, and when $T < T_c$, the number of particles of the excited states, which contributes to the heat capacity, varies with temperature. Using equation (23), one can obtain the heat capacity at a fixed external potential as

$$\frac{C_d}{Nk} = \begin{cases} \lambda(\lambda + 1)g_{\lambda+1}(z)/g_{\lambda}(z) - \lambda^2 g_{\lambda}(z)/g_{\lambda-1}(z) & T > T_c \\ \lambda(\lambda + 1)\zeta(\lambda + 1)/\zeta(\lambda) & T \leqslant T_c. \end{cases} \tag{27}$$

From equation (27) the jump in the heat capacity at the critical temperature

$$\left[\frac{\Delta C_d}{Nk}\right]_{T=T_c} = \lambda^2 \frac{\zeta(\lambda)}{\zeta(\lambda - 1)} \tag{28}$$

can be calculated. The method mentioned above may similarly be used for the case of a spherical symmetric external potential. We can obtain a series of the corresponding results, as long as the parameters $\alpha$ and $\lambda$ in (22)–(28) are replaced by $\alpha_0$ and $\lambda_0$ [8], respectively.

According to equations (25) and (28), we can obtain two new criteria:

(i) The criterion for BEC occurence

$$\lambda > 1. \tag{29}$$

(ii) The criterion on the discontinuity of the heat capacity at the critical temperature

$$\lambda > 2. \tag{30}$$

For a spherial symmetric external potential, equations (29) and (30) become $\lambda_0 > 1$ and $\lambda_0 > 2$ [8]. Equations (29) and (30) indicate the dependance of BEC upon the dimensionality of space, the kinematic characteristics of the particles, and the shapes of the external potentials. For example, when a two-dimensional non-relativistic Bose gas trapped in a spherial symmetric harmonic external potential is considred, $n = 2$, $s = 2$ and $t = 2$, so $\lambda_0 = 2 > 1$. It can be seen from criteria (i) and (ii) that BEC may occur in such a system, while the heat capacity at the critical temperature is continuous. This differs from the case of a two-dimensional non-relativistic Bose gas in the absence of an external potenetial, in which BEC cannot occur.

BEC of a three-dimension non-relativistic Bose gas trapped in a power-law potential was discussed in [9, 10], and some significant results were obtained. It is of interest to note that the results without interaction obtained in [9, 10] may be derived directly from the present paper as long as $n = 3$ and $s = 2$ are chosen. This shows that the results obtained here are more general and may be used to discuss the BEC of Bose systems with different space dimensionality and different kinematic characteristics. As an example, we consider a three-dimension non-symmetric harmonic potential and choose $U(r) = (m/2)\omega_\perp^2 r_\perp^2 + (m/2)\omega_z^2 r_z^2$. Then, if the experimental data in [6] are adopted, i.e. $N = 40\,000$, $\omega_z = 2343.63$ ($\mathrm{s}^{-1}$), and $\omega_\perp = \omega_z/8^{1/2}$, we obtain $T_\mathrm{c} = 288$ nK for non-relativistic Bose systems. This theoretical result is a very close approximation to the experimental result, $T_\mathrm{c} = 271$ nK, given in [6].

## 4. The thermodynamic properties of a degenerate Fermi gas

The fugacity $z$ in the Fermi system is not restricted. It may become very large at low temperatures. In order to obtain the Fermi integral as a quickly convergent series, we may introduce the variable $\ln z = \mu/(kT)$ to replace $z$ and use the Sommerfeld lemma [11] to expand the Fermi integral as a series:

$$f_x(z) = \frac{(\ln z)^x}{\Gamma(x+1)}\left[1 + x(x-1)\frac{\pi^2}{6}\frac{1}{(\ln z)^2} + x(x-1)(x-2)(x-3)\frac{7\pi^4}{360}\frac{1}{(\ln z)^4} + \cdots\right]. \tag{31}$$

At low temperatures, substituting the first two terms in equation (31) into equations (15), (20) and (21) gives

$$\mu = \mu_0\left[1 - (\lambda - 1)\frac{\pi^2}{6}\left(\frac{kT}{\mu_0}\right)^2\right] \tag{32}$$

$$\frac{E}{N} = \frac{\lambda}{\lambda+1}\mu_0\left[1 + (\lambda+1)\frac{\pi^2}{6}\left(\frac{kT}{\mu_0}\right)^2\right] \tag{33}$$

and

$$\frac{S}{Nk} = \frac{\lambda\pi^2}{3\mu_0}kT \tag{34}$$

where the Fermi energy

$$\mu_0 = \left[\frac{N\Gamma(\lambda+1)}{\alpha\Gamma(n/s+1)}\right]^{1/\lambda} \tag{35}$$

is closely dependent on the space dimensionality of the system, the shapes of the external potentials, and the kinematic characteristics of the particles. From equations (33) and (34) we obtain the heat capacity as

$$\frac{C}{Nk} = \frac{\lambda\pi^2}{3\mu_0}kT. \tag{36}$$

It can clearly be seen from equations (34) and (36) that both the heat capacity and entropy of the Fermi system are the same at low temperatures. This implies that they are all proportional to the temperature. This is a common characteristic which is independent of the space dimensionality, kinematic characteristics of the particles, and shapes of the external potentials, while only the proportionality coefficient depends on these parameters. The properties of the ideal Fermi gas trapped in a spherial symmentric external potential was discussed in [12]. The results obtained there may be included in the unified description of the present paper.

## 5. Thermodynamic properties of weakly degenerate and non-degenerate ideal systems

When the temperature of the system rises, the value of $z$ decreases and becomes very small. Substituting the first two terms in equations (18) and (19) into (17), one obtains

$$B \approx 1 \pm \frac{z}{2^{\lambda+1}}. \tag{37}$$

Substituting equation (37) into (14), (20) and (21), we obtain the thermodynamic functions of weakly degenerate Fermi and Bose systems as

$$q = N\left(1 \pm \frac{z}{2^{\lambda+1}}\right) \tag{38}$$

$$\frac{E}{N} = \lambda kT\left(1 \pm \frac{z}{2^{\lambda+1}}\right) \tag{39}$$

and

$$\frac{S}{Nk} = (\lambda+1)\left(1 \pm \frac{z}{2^{\lambda+1}}\right) - \ln z \tag{40}$$

where the positive sign in (37)–(40) holds for the Fermi system, while the negative sign holds for the Bose system. In the limit of high temperatures, equation (37) becomes

$$B_{\text{Fermi}} = B_{\text{Bose}} = B_{\text{classical}} = 1. \tag{41}$$

In such a case, a quantum system tends to a classical system. Hence, using the above results, we can derive the thermodynamic properties of a classical system to be

$$q = N \tag{42}$$

$$\frac{E}{N} = \lambda kT \tag{43}$$

and

$$\frac{S}{Nk} = \lambda + 1 - \ln z \tag{44}$$

where

$$z = \frac{N}{\alpha\Gamma(n/s+1)(kT)^{\lambda}} \tag{45}$$

can be derived from (15).

## 6. The density of states and the phase transition

It can be seen from equations (7) and (9) that whether particles are constrained by a spherial symmetric or non-symmetric external potential, the density of states depends only on the Hamiltonian of a single particle, while it is independent of the distribution characteristics of the particles. The density of states of particles for an $n$-dimension non-relativistic ideal system without the restriction of an external potential can be derived from (7) to be

$$D(\varepsilon) = \frac{gV_n(2\pi m)^{n/2}}{h^n\Gamma(n/2)}\varepsilon^{n/2-1} \tag{46}$$

where $V_n$ is the volume of an $n$-dimensional container and $m$ is the mass of a particle. It can be seen from (46) that when other conditions are the same, the density of low-energy excited states depends on the space dimensionality. For a three-dimensional system, $D(\varepsilon) \propto \varepsilon^{1/2}$. When $\varepsilon \to 0, D(\varepsilon) \to 0$. Thus, the thermo-fluctuations at low temperatures are very small, so that the system may be kept in long-range order. Only when the temperature rises may the thermo-fluctuations destroy long-range order and the phase transition occur. For a one-dimensional system, the density of low-energy excited states $D(\varepsilon) \propto 1/\varepsilon^{1/2}$. When $\varepsilon \to 0, D(\varepsilon) \to \infty$. Thus, no matter how low the temperature is, the fluctuations resulting from the thermo-excitation are all very strong, so that the system cannot be kept in order and the phase transition cannot take place. For a two-dimensional system, the density of states is constant and there always exists a certain thermo-fluctuation. Thus, in general, long-range order does not exist and there is only quasi-long-range order for two-dimensional systems. This shows that the density of states of particles for different space dimensionalities are different from each other, so that the properties of the different dimensional systems are different. For example, BEC may occur in three-dimensional free Bose systems, but may not occur in one- and two-dimensional free Bose systems.

When an external potential is present, the $D(\varepsilon) \sim \varepsilon$ relation will be changed. For example, for a spherial symmetric harmonic external potential, using equation (9) we can obtain the density of states for an $n$-dimension non-relativistic ideal system as

$$D(\varepsilon) = \frac{g}{(\hbar\omega)^n\Gamma(n)}\varepsilon^{n-1} \tag{47}$$

where $\omega$ is the frequency of the harmonic external potential. It can be seen from (47) that for three- and two-dimensional systems, the densities of states are proportional to $\varepsilon^2$ and $\varepsilon$, respectively. When $\varepsilon \to 0$, $D(\varepsilon) \to 0$, so that the thermo-fluctuations are very small at low temperatures, the system may be kept in long-range order. It is thus obvious that BEC may occur in three- and two-dimensional Bose systems trapped in a harmonic external potential, while it may not occur in one-dimensional systems. This is compatible with (25). It can also be seen from equations (46) and (47) that the density of states is affected by not only the space dimensionality but also the restriction of the external potential. For a three-dimensional system, the existance of the external potential makes $D(\varepsilon)$ tend to zero more quickly as $\varepsilon \to 0$. This shows that the system contrained by an external potential may be kept in long-range order more easily than a free system, such that can the phase transition occurs at a higher temperature for such a system. This conclusion has been proved by experiments. In fact, it is precisely the restriction of the suitable external potential used by three groups [3–5] in America to achieve BEC in 1995.

## 7. Conclusions

In the present paper, the conventional method of statistical physics is used to calculate the energy spectrum, density of states and grand potential of the system. Consequently, the unified description of the thermodynamic proporties of a class of systems is given by using only a few formulae. This description enables us to know quickly how the properties of a system depends on the space dimensionality, the kinematic charateristics of the particles, the shapes of the external potentials, and the particle distribution characteristics, and acquire a deep understanding of the properties of the material. In particular, for a degenerate Bose gas trapped in an external potential, two important criteria concerning BEC are given. It is of considerable importance for a deep understanding of the properties of the quantum gas to have revealed the common characteristics of the heat capacity and entropy of the Fermi system at low temperatures.

Although only particles trapped in the external potentials have been studied, the results obtained here may be used to describe a free system. Because, when $t_i \to \infty$, $U \to \infty$ and $U \to 0$ in the regions $|r_i| > L_i$ and $|r_i| < L_i$, respectively. This is just the condition of a free system confined in an $n$-dimensional container with a side length $2L_i$. Using this condition, we may obtain the density of states for a free system as

$$D(\varepsilon) = \frac{g}{h^n}\frac{n}{s}C_n V_n \frac{{p_0}^n}{\varepsilon_0^{n/s}}\varepsilon^{n/s-1}. \tag{48}$$

This is simply one of the main results in [2]. Substituting equation (48) into (12), we can derive the corresponding grand potential, from which one can find all the thermodynamic properties of free ideal systems with different space dimensions and different particle kinematic characteristics. For example, from equation (14) one can obtain the unified equation of states of free ideal systems as

$$\frac{P_n V_n}{kT} = NB \tag{49}$$

where $P_n$ is the pressure of an $n$-dimensional free system. In particular, when $n = 3$, $s = 2$, and $\varepsilon_0/P_0^s = 1/2m$, we can derive all the results of the ideal Bose, Fermi and classical systems, as given in statistical physics textbooks. This shows once again that the unified description obtained in the present paper is of considerable importance.

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